An example of Feynman-Jackson integral

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Abstract

We review the construction of a q-analogue of the Gaussian measure. We apply that construction to obtain a q-analogue of Feynman integrals and to compute explicitly an example of such integrals.

1 Introduction

The main goal of this note is to provide a gentle introduction to the theory of Feynman-Jackson integrals, and to compute an example of such integrals as explicitly as possible. Roughly speaking Feynman-Jackson integrals are the analogue in q-calculus of Feynman integrals. The name of Jackson is included since integration in q-calculus was introduced by F.H. Jackson, see [9] and [10]. Our computations are done in the one dimensional setting. Extending our concepts to higher dimensions, in particular to infinite dimensions, is the main problem in this subject. In principle such extension is possible but making the notation easy to handle is hard, as the reader will learn by looking at the simple case of one dimensional integrals.

The other fundamental task in this subject matter is to find applications of Feynman-Jackson integrals in physics. Although in this note we focus on the mathematical properties of such integrals, we believe that our formalism will find applications thanks to the following facts:

- In this paper our emphasis is on Feynman-Jackson integrals which rely on the construction of a q-analogue of the Gaussian measure. Given the widespread range of applications in mathematics and physics alike of the Gaussian measure, we expect its q-analogue to gradually find its natural set of applications. The subject of q-probability theory and q-random processes is still in a developing phase, but already a solid step forward has been taken by Kupershmidt in [12].
- q-calculus is adapted to work with arbitrary functions while the usual rules of calculus demand certain kind of regularity. The q-Gaussian measure is likely to find applications in the context of highly non-regular phenomena.

- Jackson integrals are given by infinite sums. Cutting off the number of terms appearing in such sums provides a natural regularization method for diverging Jackson integrals.

- Classically, we think of the continuous as a limit of the discrete. Indeed, in the limit $q \to 1$ we recover from Feynman-Jackson integrals the usual Feynman integrals. Thus our Feynman-Jackson integrals provide a new method for computing Feynman integrals: first compute the corresponding Feynman-Jackson integral and then take the limit as q goes to 1.

- From a quantum perspective, it is the continuous that should be regarded as an approximation to the fundamental discrete quantum theory. In particular it has been argued, see [13], [14], that in the quantization of gravity discrete structures will emerge naturally. If that is so, then integration over discrete structures may become a fundamental issue. Our formalism may shed some light as the form that such theory of discrete integration may take. We emphasize that the calculus itself in q-calculus is discrete while the variables involved remain continuous.

2 Gauss-Jackson integrals

Let us recall some notions of q-calculus, see [1], [2],[6] and [8] for more information. Fix a real number $0 < q < 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $x \in \mathbb{R}$ be a real number. The q-derivative of f at x is given by

$$\partial_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1}$$

For example if $t \in \mathbb{R}$ then $\partial_q x^t = [t]_q x^{t-1}$ where $[t]_q = \frac{q^t - 1}{q-1}$.

The q-integral, better known as the Jackson integral, of f between 0 and $b \in \mathbb{R}^+$ of f is given by

$$\int_0^b f(x) d_q x = (1-q)b \sum_{n=0}^{\infty} q^n f(q^n b). \tag{2}$$

We also define

$$\int_{-b}^0 f(x) d_q x = \int_0^b f(-x) d_q x \tag{3}$$

and

$$\int_{-b}^b f(x) d_q x = \int_{-b}^0 f(x) d_q x + \int_0^b f(x) d_q x. \tag{4}$$

Notice that in the limit $q \to 1$ the q-derivative and the q-integral approach the usual derivative and the Riemann integral, respectively. The q-analogues of the rules of derivation and

integration by parts are

$$\partial_q(fg)(x) = \partial_q f(x)g(x) + f(qx)\partial_q g(x), \tag{5}$$

$$\int_0^b \partial_q f(x)g(x)d_qx = -\int_0^b f(qx)\partial_q g(x)d_qx + f(b)g(b) - f(0)g(0). \tag{6}$$

The first goal of this note is to describe the q-analogue of the Gaussian measure on $\mathbb{R}$. The moments of the Gaussian measure are given by the integrals

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}x^n dx. \tag{7}$$

A remarkable property of the Gaussian measure is that it provides a bridge between measure theory and combinatorics. Indeed, the moments of the Gaussian measure are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}x^{2n} dx = (2n-1)(2n-3)...7.5.3.1, \tag{8}$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}x^{2n+1} dx = 0. \tag{9}$$

The number $(2n-1)(2n-3)...7.5.3$ is often denoted by $(2n-1)!!$ and is called the double factorial. The reader may consult [3] for a natural generalization of such numbers. It can be shown that $(2n-1)(2n-3)...7.5.3$ counts the number of pairings on the set $[2n] = \{1, 2, ..., 2n\}$. A pairing on $[2n]$ is a partition of $[2n]$ into n blocks each of cardinality two. So for example we have that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}x^4 dx = 3, \tag{10}$$

since as shown in Figure 3 there are 3 pairings on a set with 4 elements.

Thus we see that the Gaussian measure has a clear cut combinatorial meaning. This simple fact explains the source of graphs in the computation of Feynman integrals. In order to define the q-analogue of the Gaussian measure we must find q-analogues for the objects appearing in the Gaussian measure, namely $\sqrt{2\pi}, \infty, e^{-\frac{x^2}{2}}, x^n$ and dx. The Lebesgue measure dx agrees with Riemann integration for good functions. Thus it is only natural to replace dx by Jackson integration d_qx. While the factor x^n remains unchanged, finding the q-analogue of $e^{-\frac{x^2}{2}}$ is actually quite a subtle matter. First, we must find a q-analogue for the exponential function e^x which is characterized by the properties $\partial e^x = e^x$ and $e^0 = 1$. So we look for a function e_q^x such that $\partial e_q^x = e_q^x$ and $e_q^0 = 1$. A solution to this couple of equations is

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{11}$$

where

$$[n]_q! = [n]_q[n-1]_q[n-2]_q...[2]_q \quad \text{and} \quad [n]_q = 1 + q + q^2 + q^3 + ... + q^{n-1}.$$

The q-analogue of the identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where

$$E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}. \tag{12}$$

Once we have obtained q-analogues for the exponential map and its inverse one might think that it is straightforward to generalized the term $e^{-\frac{x^2}{2}}$ of the Gaussian integrals. However, this is not the case and while our first impulse is to try $E_q^{-\frac{x^2}{[2]_q}}$, the right answer [5] is to replace $e^{-\frac{x^2}{2}}$ by

$$E_{q^2}^{-\frac{q^2x^2}{[2]_q}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} x^{2n}}{(1+q)^n [n]_{q^2}!}. \tag{13}$$

Now we consider the integration limits. It is amusing that whereas the classical Gaussian measure is given by an improper integral, its q-analogue turns out to be a definite integral whose limits depend on q and go to (plus or minus) infinity as q approaches to 1. A similar situation occurs with the integral representations of the q-analogue of the gamma function, see [11] and [15]. Indeed, without further motivation we shall take the boundary limits in the Gaussian integrals to be $-\nu$ and ν where $\nu = \frac{1}{\sqrt{1-q}}$. To find the q-analogue $c(q)$ of the $\sqrt{2\pi}$ appearing in Gaussian integrals we must demand that $c(q)$ be such that

$$\frac{1}{c(q)} \int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} d_q x = 1. \tag{14}$$

Thus $c(q)$ is given by

$$c(q) = \int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} d_q x = 2\int_{0}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} d_q x = 2(1-q)\nu \sum_{n=0}^{\infty} q^n E_{q^2}^{\frac{-q^2(q^n\nu)^2}{[2]_q}}, \tag{15}$$

so

$$c(q) = 2\sqrt{1-q} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)+(2m+1)n}}{(1-q^2)^m [m]_{q^2}!} \tag{16}$$

and interchanging the order of summation we get the identity

$$c(q) = 2\sqrt{1-q} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)}}{(1-q^{2m+1})(1-q^2)^m [m]_{q^2}!}. \tag{17}$$

Since $\mathrm{Lim}_{q\to 1} c(q) = \sqrt{2\pi}$ we obtain the amusing identity

$$\sqrt{2\pi} = 2\mathrm{Lim}_{q\to 1}\sqrt{1-q}\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)}}{(1-q^{2m+1})(1-q^2)^m[m]_{q^2}!}. \tag{18}$$

We now look for the q-analogue of identities (8) and (9) for the moments of Gaussian integrals. A key result is that one can show the identities

$$\frac{1}{c(q)}\int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} x^{2n} d_q x = [2n-1]_q[2n-3]_q...[7]_q[5]_q[3]_q[1]_q = [2n-1]_q!!, \tag{19}$$

and

$$\frac{1}{c(q)}\int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} x^{2n+1} d_q x = 0. \tag{20}$$

Identity (20) follows from the fact that x^{2n+1} is an odd function and $E_{q^2}^{\frac{-q^2x^2}{[2]_q}}$ is an even function. Identity (19) is proved recursively. Using formula (6), one shows that

$$\frac{1}{c(q)}\int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} x^{2n+2} d_q x = \frac{[2n+1]_q}{c(q)}\int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} x^{2n} d_q x. \tag{21}$$

Identities (14) and (21) imply identity (19). Next we describe a combinatorial interpretation of the number

$$[2n-1]_q!! = [2n-1]_q[2n-3]_q...[7]_q[5]_q[3]_q[1]_q. \tag{22}$$

An ordered pairing p on $[2n] = \{1, 2, ..., 2n\}$ is a sequence $p = \{\ (a_i, b_i)\ \}_{i=1}^{n} \in ([2n]^2)^n$ such that

- $a_1 < a_2 < \cdots < a_n$.
- $a_i < b_i, \quad i = 1, \ldots, n$.
- $[2n] = \bigsqcup_{i=1}^{n}\{a_i, b_i\}$.

We denote by $P[2n]$ the set of ordered pairings on $[2n]$. We are going to define a weight $w(p)$ for each $p \in P[2n]$. Let us introduce the following notation

- $((a_i, b_i)) = \{j \in [[2n]] : a_i < j < b_i\}$ for all $(a_i, b_i) \in p$.
- $B_i(p) = \{b_j : 1 \leq j < i\}$.

The weight of p is defined by the rule

$$w(p) = \prod_{i=1}^{n} q^{|((a_i,b_i))\setminus B_i(p)|} = q^{\sum_{i=1}^{n}|((a_i,b_i))\setminus B_i(p)|}. \tag{23}$$

Figure 1 and Figure 2 below show a couple of examples of pairings together with the corresponding weights

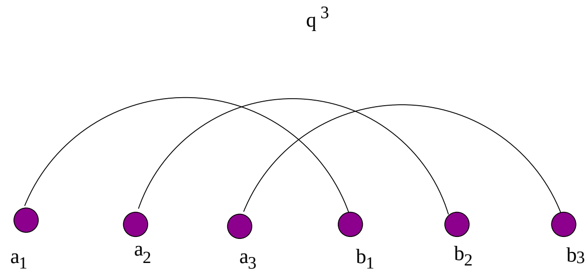

Figure 1: A Pairing with weight q^3.

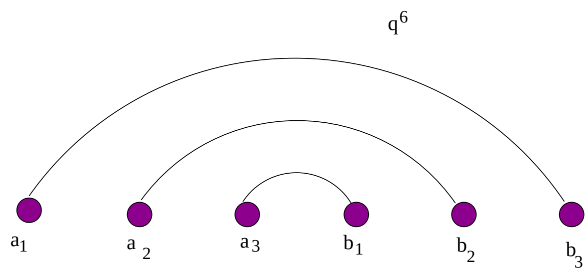

Figure 2: A Pairing with weight q^6.

Using this language we can state the following result which is proved by induction

$$[2n-1]_q!! = \sum_{p \in P[2n]} w(p). \tag{24}$$

Notice that as $q \longrightarrow 1$ we recover from (24) the well-known identity

$$(2n-1)!! = |\{\text{pairings on } [[2n]]\}|.$$

For example $[3]_q = 1 + q + q^2$ which agrees with the sum of the weights of the three pairings on [4] shown in Figure 3.

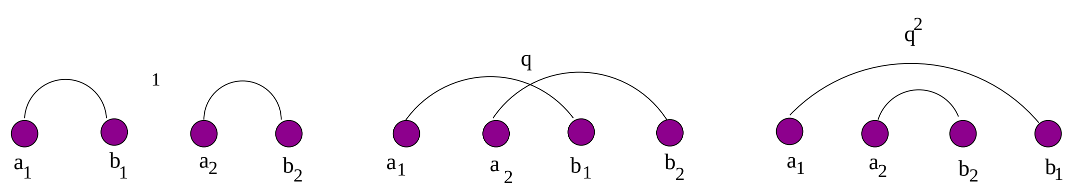

Figure 3: Combinatorial interpretation of $[3]_q$.

In conclusion we have proved that

$$\frac{1}{c(q)} \int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q}} x^{2n} d_q x = \sum_{p \in P[2n]} w(p). \tag{25}$$

3 Examples of Feynman-Jackson integrals

We want to study the computation of Feynman-Jackson integrals of the form

$$I(g) = \frac{1}{c(q)} \int_{-\nu}^{\nu} E_{q^2}^{\frac{-q^2x^2}{[2]_q} + g\frac{x^3}{[3]_q!}} d_q x. \tag{26}$$

As usual in the theory of Feynman integrals we regard $I(g)$ as a formal power series in the formal variable g, i.e., $I(g) \in \mathbb{C}[[g]]$. The first step in the computation of a Feynman-Jackson integrals is to reduce it to the computation of a countable number of Gaussian-Jackson integrals. This step is carry out with the help of following formula proved in [4]

$$E_{q^2}^{x+y} = E_{q^2}^{x} \left(\sum_{c,d \geq 0} \lambda_{c,d} x^c y^d \right) \tag{27}$$

where $\lambda_{c,d} = \sum_{k=0}^{c} \frac{(-1)^{c-d} \binom{d+k}{k} q^{(d+k)(d+k-1)}}{[d+k]_{q^2}! [c-k]_{q^2}!}$. Making the substitutions $x \rightarrow -\frac{q^2x^2}{[2]_q}$ and $y \rightarrow g\frac{x^3}{[3]_q!}$ we obtain

$$E_{q^2}^{-\frac{q^2x^2}{[2]_q} + g\frac{x^3}{[3]_q!}} = E_{q^2}^{-\frac{q^2x^2}{[2]_q}} \sum_{c,d \geq 0} \lambda_{c,d} \frac{(-1)^c q^{2c} x^{2c} x^{3d}}{[2]_q^c ([3]_q!)^d} g^d = E_{q^2}^{-\frac{q^2x^2}{[2]_q}} \sum_{c,d \geq 0} \lambda_{c,d} \frac{(-1)^c q^{2c} x^{2c+3d}}{[2]_q^c ([3]_q!)^d} g^d, \tag{28}$$

so we get

$$E_{q^2}^{-\frac{q^2x^2}{[2]_q} + g\frac{x^3}{[3]_q!}} = \sum_{c,d,k} \frac{(-1)^{2c-k} \binom{d+k}{k} q^{(d+k)(d+k-1)+2c}}{[2]_q^c ([3]_q!)^d [d+k]_{q^2}! [c-k]_{q^2}!} x^{2c+3d} g^d. \tag{29}$$

If we q-integrate both sides of equation (29) we get

$$\begin{aligned} \frac{1}{c(q)} \int_{-v}^{v} E_{q^2}^{-\frac{q^2x^2}{[2]_q} + g\frac{x^3}{[3]_q!}} d_q x &= \sum_{c,d,k} \frac{(-1)^{2c-k} \binom{2d+k}{k} q^{(2d+k)(2d+k-1)+2c} [2c+6d-1]_q!!}{[2]_q^c ([3]_q!)^{2d} [d+k]_{q^2}! [c-k]_{q^2}!} g^{2d} &(30) \\ &= \sum_{c,d,k} \frac{(-1)^{2c-k} \binom{2d+k}{k} q^{(2d+k)(2d+k-1)+2c} [2c+6d-1]_q!!}{[2]_q^c ([3]_q!)^{2d} [2d+k]_{q^2}! [c-k]_{q^2}!} g^{2d} &(31) \end{aligned}$$

The second step in the computation of a Feynman integral is to write such as integral as a sum of a countable number of contributions, where each summand is naturally associated to certain kind of graph, see [7]. So we want to understand the right hand side of the equation (31) in terms of a summation of the weights of an appropriated set of isomorphism classes of graphs. Consider the category $\mathbf{Graph}_q^3$ whose objects are planar graphs (V, E, b) such that

1. $V = \{\bullet\} \sqcup V_1 \sqcup V_2$, where $V^1 = \{\otimes_1, \ldots, \otimes_c\}$ and $V^2 = \{\circ_1, \ldots, \circ_d\}$.
2. $E = E_1 \sqcup E_2 \sqcup E_3$.
3. $b : E \longrightarrow P_2(V) = \{A \subset V |\ 1 \leq |A| \leq 2\}$.

We use the following notation

$$I(v,e) = \begin{cases} 0, & \text{if } v \notin b(e) \\ 1, & \text{if } |b(e)| = 2, v \in b(e) \\ 2, & \text{if } |b(e)| = \{v\} \end{cases}$$

This data must satisfy the following axioms:

1. $\sum_{e \in E_3} I(\otimes_i, e) = 2$ and $\sum_{e \in E_3} I(\circ_j, e) = 3$.
2. $|b^{-1}(\otimes_i, \bullet)| \leq 1$ and if $|b^{-1}(\otimes_i, \bullet)| = 1$ then $|b^{-1}(\otimes_j, \bullet)| = 1$ for $i \leq j \leq c$.
3. If $e \in E_1 \sqcup E_2$, then $b(e) = \{\otimes_i, \bullet\}$ or $b(e) = \{\circ_j, \circ\}$ for some $1 \leq i \leq c$ or$1 \leq j \leq d$.
4. If $e \in E_3$ then $\bullet \notin b(e)$.
5. $|E_2| \leq |V_1|$.

   To each graph $\Gamma = (V, E, b)$ as above we associate two polynomials in q, $\omega_q(\Gamma)$ and $a_q(\Gamma)$, given respectively, by

   (a) $\omega_q(\Gamma) = (-1)^{|E_1|} q^{2|V_1| + 2\binom{|V_2| + |E_2|}{2}} \omega(p)$. Above p is the natural pairing induced by E_3 on the flags of Γ, i.e.,the set $\{(\otimes_i, e) |\ \otimes_i \in e\} \sqcup \{(\circ_j, e) |\ \circ_i \in e\}$. $\omega(p)$ is the weight of p as given by formula (23).

   (b) $a_q(\Gamma) = [2]_q^{|V_1|} ([3]_q!)^{|V_2|} [|V_2| + |E_2|]_{q^2}! [|V_1| - |E_2|]_{q^2}!$.

Figure 4 and Figure 5 show examples of graphs in $\mathbf{Graph}_q^3$ together with the corresponding polynomials a_q and ω_q.

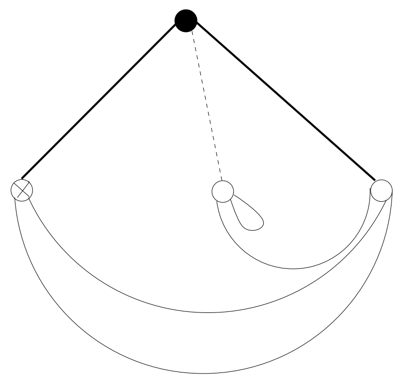

Figure 4: Feynman graph Γ with $\omega_q(\Gamma) = -q^{16}$ and $a_q(\Gamma) = [2]_q^4[3]_q^3$.

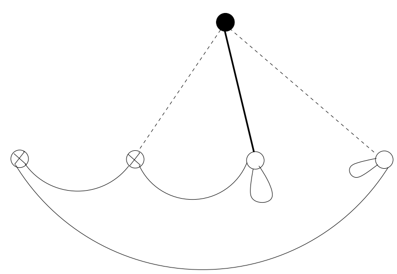

Figure 5: Feynman graph Γ with $\omega_q(\Gamma) = q^{22}$ and $a_q(\Gamma) = [2]_q^2([3]_q!)^2[4]_q!$.

Now we are ready to state the main result of this paper, namely, we write the integral (26) as a sum over the weights of graphs as follows

$$\frac{1}{c(q)}\int_{-v}^{v} E_{q^2}^{-\frac{q^2x^2}{[2]_q}+g\frac{x^3}{[3]_q!}} d_qx = \sum_{(V,E,b)\in\mathbf{Graph}_q^3/\sim} \frac{\omega_q(\Gamma)}{a_q(\Gamma)} g^{|V_2|} \tag{32}$$

where the sum runs over all isomorphisms classes of graphs in $\mathbf{Graph}_q^3$. Identity (32) follows directly from identity (31) and the definitions above.

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